Heavy quark potential in the static limit of QCD

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Following the procedure and motivations developed by Richardson, Buchmüller, and Tye, we derive the potential of static quarks consistent with both the three-loop running of the QCD coupling constant under the two-loop perturbative matching of V and $\overline{\mathrm{MS}}$ schemes and the confinement regime at long distances. The implications for the heavy quark masses as well as the quarkonium spectra and leptonic widths are discussed.

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I. INTRODUCTION

The potential of static heavy quarks illuminates the most important features of QCD dynamics: the asymptotic freedom and confinement. Trying to study subtle electroweak phenomena in the heavy quark sector of the standard model, we need quite an accurate quantitative understanding of the effects caused by the strong interactions. In addition to the perturbative calculations for hard contributions, at present there are three general approaches to get a systematic description of how the heavy quarks are bound to the hadrons and what the relations are between the measured properties of such hadrons and the characteristics of heavy quarks as relevant to the electroweak interactions and QCD. These approaches are the operator product expansion (OPE) in the inverse powers of heavy quark mass, the sum rules (SR's) of QCD, and the potential models for the systems containing the heavy quarks by exploring various approximations of the Bethe-Salpeter equation with the static potential treated in the framework of effective theory with a power counting in terms of powers of the inverse heavy quark mass. The first method is usually exploited in the inclusive estimates, while the second and third techniques are the frameworks of exclusive calculations. The important challenge is a consistency of evaluations obtained in such ways that requires the comparative analysis of calculations. A wide variety of systems and processes for the analysis provides a more complete qualitative and quantitative understanding of heavy quark dynamics.

In the leading order of perturbative QCD at short distances and with a linear confining term in the infrared region, the potential of static heavy quarks was considered in the Cornell model [1], incorporating the simple superposition of both asymptotic limits (the effective Coulomb and string-like interactions). The observed heavy quarkonia posed in the intermediate distances, where both terms are important for the determination of mass spectra. So, the phenomenological approximations of potential (logarithmic one [2] and power law [3]), taking into account the regularities of such the spectra, were quite successful [4], while the quantities more sensitive to the global properties of potential are the wave functions at the origin as related to the leptonic constants and production rates. So, the potentials consistent with the asymptotic freedom to one and two loops as well as the linear confinement were proposed by Richardson [5], and Buchmüller and Tye [6], respectively. Technically, using a given scheme of regularization, say, the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$), one has to calculate the perturbative expansion for the potential of static quarks. This potential can be written down as the Coulomb one with the running coupling constant in the so-called V scheme. Thus, the perturbative calculations provide us with the matching of $\overline{\mathrm{MS}}$ scheme with the V scheme. The n loop running of $\alpha_s^{\overline{\mathrm{MS}}}$ requires the $n-1$ loops to match α_V. Note, that initially two coefficients of corresponding β functions are scheme and gauge independent, while the others are generally dependent. With the dynamical fields integrated out, the V scheme is defined in terms of the action depending on the static sources (the distance r), so that its β function is gauge invariant. The motivation by Buchmüller and Tye was to write down the β function of α_V consistent with two known asymptotic regimes at short and long distances. They proposed the function, which results in the effective charge determined by two parameters, only: the perturbative parameter is the scale in the running of coupling constant at large virtualities and the nonperturbative parameter is the string tension. The necessary inputs are the coefficients of β function. Two loop results and one loop matching condition were available for the BT model. Recently, the progress in calculations has provided us with the two loop matching of V and $\overline{\mathrm{MS}}$ schemes [7,8], which can be combined with the three loop running of $\alpha_s^{\overline{\mathrm{MS}}}$. Therefore, the modification of Buchmüller-Tye (BT) potential of static quarks as dictated by the current status of perturbative calculations is of great interest. Moreover, at the moment two peculiar questions become apparent. First, the asymptotic perturbative expansion of the BT β function to the third order results in the three loop coefficient, which is wrong even in its sign. Second, the elaborated $\Lambda_{\overline{\mathrm{MS}}}$ parameter by BT is in a deep contradiction with the measured value

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[9]. To clarify the situation, we will derive the static quark potential consistent from the state of the art.

Thus, our motivation is to combine high order multiloop calculations of the perturbative static potential [7,8] with the string tension ansatz. We improve the perturbative input for the potential model in order to remove the contradiction between the modern high energy data on the QCD coupling constant and the description of heavy quark potential in the framework of the one-loop Buchmüller-Tye model, which accepts an extremely high value of the coupling constant evolved to the Z mass scale. In other words, if we accept the current normalization of the coupling constant and introduce its value into the Buchmüller-Tye approach to the one-loop potential, then we get the contradiction of such the potential with the heavy quarkonium mass spectra, certainly, since we find about 200 MeV smaller splitting between the $1S$ and $2S$ levels in comparison with the experimental 580 MeV. This discrepancy cannot be removed by the modification of the nonperturbative part in the potential with no contradiction with the data on the slope of Regge trajectories. Therefore, the modification of perturbative input for the model of static potential in QCD is meaningful in this sense even when the nonperturbative contribution is conserved in the old string tension form. So, the significant improvement of the perturbative β function for the charge in the Coulomb potential is combined with the consequent evolution from high virtualities to low ones, taking into account the influence of the nonperturbative term on the evolution, which becomes essential numerically below the scale of 4 GeV.

We have to emphasize that at the moment the paper by Buchmüller and Tye [6] was published, a theory for the heavy nonrelativistic $Q\bar{Q}$ pair did not exist. So, the phenomenological derivation of the static potential including perturbative short-distance and nonperturbative long-distance elements made by BT was all one could do. At present, at least for very heavy quarks, such a theory does exist in the form of potential nonrelativistic QCD (PNRQCD) [10] and velocity-counting nonrelativistic QCD (vNRQCD) [11], and we address the comparison of the static potential model developed in this work with these sound theoretical approaches in QCD to the physics of heavy quarkonium.

Another aspect of this work is devoted to the heavy quark masses. After the potential is given, the heavy quark masses incorporated in the corresponding Schrödinger equation determine the heavy quarkonium spectra with no ambiguity.[1] These masses involved in the potential model are denoted by m_Q^V. This mass should be distinguished from the pole mass which is a purely perturbative concept defined unambiguously at each order of perturbation theory through the pole of the perturbative heavy quark propagator. Thus, we need to test the consistency of estimates for the masses in the QCD potential of static quarks and in SR.

In Sec. II we generalize the BT approach to three loops and derive the static potential of heavy quarks. The numerical values of potential parameters and their consistency with the relevant quantities are considered. The implications for the heavy quark masses, spectra of heavy quarkonia, and leptonic constants are discussed in Sec. III. The obtained results are summarized in the conclusions (Sec. IV).

[1]We deal with the so-called spin-averaged spectra, since the consideration of spin-dependent splitting involves some additional parameters beyond the static potential.

II. QCD AND POTENTIAL OF STATIC QUARKS

In this section, first, we discuss two regimes for the QCD forces between the static heavy quarks: the asymptotic freedom and confinement. Second, we formulate how they can be combined in a unified β function obeying both limits of small and large QCD couplings.

A. Perturbative results at short distances

The static potential is defined in a manifestly gauge invariant way by means of the vacuum expectation value of a Wilson loop [12]

$$V(r) = -\lim_{T\to\infty}\frac{1}{iT}\ln\langle\mathcal{W}_\Gamma\rangle,$$

$$\mathcal{W}_\Gamma = \widetilde{\mathrm{tr}}\,\mathcal{P}\exp\left(ig\oint_\Gamma dx_\mu A^\mu\right). \qquad (1)$$

Here, Γ is taken as a rectangular loop with time extension T and spatial extension r. The gauge fields A_μ are path ordered along the loop, while the color trace is normalized according to $\widetilde{\mathrm{tr}}(\ldots) = \mathrm{tr}(\ldots)/\mathrm{tr}\mathbb{1}$.

Generally, one introduces the V scheme of QCD coupling constant by the definition of QCD potential of static quarks in momentum space as follows:

$$V(\mathbf{q}^2) = -C_F\frac{4\pi\alpha_V(\mathbf{q}^2)}{\mathbf{q}^2}, \qquad (2)$$

while α_V can be matched with $\alpha_{\overline{\mathrm{MS}}}$

$$\alpha_V(\mathbf{q}^2) = \alpha_{\overline{\mathrm{MS}}}(\mu^2)\sum_{n=0}^\infty \tilde{a}_n(\mu^2/\mathbf{q}^2)\left(\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{4\pi}\right)^n \qquad (3)$$

$$= \alpha_{\overline{\mathrm{MS}}}(\mathbf{q}^2)\sum_{n=0}^\infty a_n\left(\frac{\alpha_{\overline{\mathrm{MS}}}(\mathbf{q}^2)}{4\pi}\right)^n. \qquad (4)$$

At present, our knowledge of this expansion[2] is restricted by

$$a_0 = \tilde{a}_0 = 1, \quad a_1 = \frac{31}{9}C_A - \frac{20}{9}T_F n_f, \quad \tilde{a}_1 = a_1 + \beta_0\ln\frac{\mu^2}{\mathbf{q}^2}, \qquad (5)$$

which is the well-known one-loop result, and the recent two-loop calculations [7,8], which gave

[2]On a possible peculiar behavior in the expansion see Ref. [12].

$$a_2=\left(\frac{4343}{162}+4\pi^2-\frac{\pi^4}{4}+\frac{22}{3}\zeta(3)\right)C_A^2-\left(\frac{1798}{81}+\frac{56}{3}\zeta(3)\right)C_AT_Fn_f-\left(\frac{55}{3}-16\zeta(3)\right)C_FT_Fn_f+\left(\frac{20}{9}T_Fn_f\right)^2, \tag{6}$$

$$\tilde{a}_2=a_2+\beta_0^2\ln^2\frac{\mu^2}{\mathbf{q}^2}+(\beta_1+2\beta_0a_1)\ln\frac{\mu^2}{\mathbf{q}^2}. \tag{7}$$

We have used here the ordinary notations for the SU(N_c) gauge group: $C_A=N_c$, $C_F=(N_c^2-1)/2N_c$, $T_F=\frac{1}{2}$. The number of active flavors is denoted by n_f.

After the introduction of $\mathfrak{a}=\alpha/4\pi$, the β function is actually defined by

$$\frac{d\mathfrak{a}(\mu^2)}{d\ln\mu^2}=\beta(\mathfrak{a})=-\sum_{n=0}^{\infty}\beta_n\cdot\mathfrak{a}^{n+2}(\mu^2), \tag{8}$$

so that $\beta_{0,1}^V=\beta_{0,1}^{\overline{\rm MS}}$ and

$$\beta_2^V=\beta_2^{\overline{\rm MS}}-a_1\beta_1^{\overline{\rm MS}}+(a_2-a_1^2)\beta_0^{\overline{\rm MS}}=\left(\frac{618+242\zeta(3)}{9}+\frac{11(16\pi^2-\pi^4)}{12}\right)C_A^3-\left(\frac{445+704\zeta(3)}{9}+\frac{16\pi^2-\pi^4}{3}\right)C_A^2T_Fn_f+\frac{2+224\zeta(3)}{9}C_A(T_Fn_f)^2-\frac{686-528\zeta(3)}{9}C_AC_FT_Fn_f \tag{9}$$

$$+2C_F^2T_Fn_f+\frac{184-192\zeta(3)}{9}C_F(T_Fn_f)^2. \tag{10}$$

The coefficients of β function, calculated in the $\overline{\rm MS}$ scheme [13], are given by

$$\beta_0^{\overline{\rm MS}}=\frac{11}{3}C_A-\frac{4}{3}T_Fn_f, \tag{11}$$

$$\beta_1^{\overline{\rm MS}}=\frac{34}{3}C_A^2-4C_FT_Fn_f-\frac{20}{3}C_AT_Fn_f, \tag{12}$$

$$\beta_2^{\overline{\rm MS}}=\frac{2857}{54}C_A^3+2C_F^2T_Fn_f-\frac{205}{9}C_AC_FT_Fn_f-\frac{1415}{27}C_A^2T_Fn_f+\frac{44}{9}C_F(T_Fn_f)^2+\frac{158}{27}C_A(T_Fn_f)^2. \tag{13}$$

The Fourier transform results in the position-space potential [7]

$$V(r)=-C_F\frac{\alpha_{\overline{\rm MS}}(\mu^2)}{r}\left(1+\frac{\alpha_{\overline{\rm MS}}(\mu^2)}{4\pi}(2\beta_0\ln(\mu r')+a_1)+\left(\frac{\alpha_{\overline{\rm MS}}(\mu^2)}{4\pi}\right)^2\left(\beta_0^2\left(4\ln^2(\mu r')+\frac{\pi^2}{3}\right)+2(\beta_1+2\beta_0a_1)\ln(\mu r')+a_2\right)+\cdots\right) \tag{14}$$

with $r'\equiv r\exp(\gamma_E)$. Defining the new running coupling constant, depending on the distance,

$$V(r)=-C_F\frac{\overline{\alpha}_V(1/r^2)}{r}, \tag{15}$$

we can calculate its β function from Eq. (14), so that [7]

$$\overline{\beta}_2^V=\beta_2^V+\frac{\pi^2}{3}\beta_0^3, \tag{16}$$

and the minor coefficients $\overline{\beta}_{0,1}^V$ are equal to the scheme-independent values given above.

To normalize the couplings, we use Eq. (4) at $\mathbf{q}^2=m_Z^2$.

B. Confining term

The nonperturbative behavior of QCD forces between the static heavy quarks at long distances r is usually represented by the linear potential (see discussion in Ref. [14])

$$V^{\rm conf}(r)=k\cdot r, \tag{17}$$

which corresponds to the square-law limit for the Wilson loop.

We can represent this potential in terms of the constant chromoelectric field between the sources posed in the fundamental representation of SU(N_c). So, in the Fock-Schwinger gauge of fixed point

$$x_\mu\cdot A^\mu(x)=0,$$

we can represent the gluon field by means of the strength tensor [15]

$$A_\mu(x)\approx-\tfrac{1}{2}x^\nu G_{\nu\mu}(0),$$

so that for the static quarks separated by the distance $\mathbf{r}$

$$\overline{Q}_i(0)G_{m0}^a(0)Q_j(0)=\frac{\mathbf{r}_m}{r}ET_{ij}^a,$$

where the heavy quark fields are normalized to unity. Then, the confining potential is written down as

$$V^{\rm conf}(r)=\tfrac{1}{2}g_sC_FE\cdot r.$$

Supposing, that the same strength of the field is responsible for the formation of gluon condensate, by introducing the colored sources n_i, which have to be averaged in the vacuum, we can easily find [16]

$$\langle G^2_{\mu\nu}\rangle = -4\langle G^a_{m0}(0)G^a_{m0}(0)\rangle = 4C_F E^2\langle \bar{n}n\rangle,$$

where we have supposed the relation

$$\langle \bar{n}T^aT^bn\rangle = -\langle \bar{n}T^an\cdot\bar{n}T^bn\rangle, \tag{18}$$

which ensures that the sources conserve the massless of the gluon, and, hence, the gauge invariance.[3] Further, it is evident that

$$\langle \bar{n}T^aT^bn\rangle = C_F\frac{\delta^{ab}}{N_c^2-1}\langle \bar{n}n\rangle.$$

Then, we conclude that the relation between the strength E and the string tension depends on the normalization of vacuum sources n_i. We put

$$\langle \bar{n}_i n_j\rangle = n_l\delta_{ij},$$

where n_l denotes the number of light *stochastic* flavors, which is the free parameter of such a representation. Of course, the value of n_l should be finite even in the case of pure gluodynamics with no light quarks in the infrared region. Moreover, the light quark loops could cause the breaking of gluon string, i.e., the strong decays of higher excitations. We assume that n_l is basically determined by the gluon dynamics (i.e., the number of colors), and it slightly correlates with the number of quark flavors. After a simple consideration of potential strength between two colored sources in the fundamental and adjoint representations, i.e., the color factors in front of a single gluon Coulomb potential, we assume that in the pure gluodynamics the number of stochastic sources substituting for the vacuum gluons can be accepted in the form[4]

$$n_l = \frac{1}{N_c}\frac{C_A}{C_F} = \frac{3}{4} = \frac{1}{4}\tilde{n}_l,$$

where the factor $1/N_c$ normalizes the source to unit, and C_A/C_F is the appropriate ratio of color charges. To the moment, the shift of n_l in QCD with light quarks is not explicitly fixed, while the lattice calculations show that the dependence of string tension on the number of light quarks is weak [17]. Finally, we find for the linear term of the potential

$$k = \frac{\pi}{\sqrt{C_F N_c \tilde{n}_l}}C_F\sqrt{\left\langle \frac{\alpha_s}{\pi}G^2_{\mu\nu}\right\rangle} = \frac{\pi}{2\sqrt{N_c}}C_F\sqrt{\left\langle \frac{\alpha_s}{\pi}G^2_{\mu\nu}\right\rangle}. \tag{19}$$

The k term is usually represented through a parameter α'_P as

$$k = \frac{1}{2\pi\alpha'_P}.$$

Buchmüller and Tye put $\alpha'_P = 1.04\ \text{GeV}^{-2}$, which we use throughout this paper. This value of tension, which is related with a slope of Regge trajectories, can be compared with the estimate following from Eq. (19). At

$$\left\langle \frac{\alpha_s}{\pi}G^2_{\mu\nu}\right\rangle = (1.6\pm0.1)\cdot10^{-2}\ \text{GeV}^4\ [18]$$

we have found

$$\alpha'_P = 1.04\pm0.03\ \text{GeV}^{-2},$$

which is in a good agreement with the fixed value.[5]

The form of Eq. (17) corresponds to the limit, when at low virtualities $\mathbf{q}^2\to0$ the coupling α_V tends to

$$\alpha_V(\mathbf{q}^2)\to\frac{K}{\mathbf{q}^2},$$

so that

$$\frac{d\alpha_V(\mathbf{q}^2)}{\ln\mathbf{q}^2}\to-\alpha_V(\mathbf{q}^2), \tag{20}$$

which gives the confinement asymptotics for the β_V function.

A special comment should be made on the role of the linear term in the potential. Considering the power corrections, which can be attributed to various sources such as the renormalon, topological effects caused by monopoles and vorteces, deviations from the operator product expansion, the authors of [19] argued that this term responsible for the quark confinement can contribute at short distances too. This conclusion is essentially different from the point of view based on the notion about a low energy phase transition leading to the condensation of gluons and quarks. This condensation provides the formation of a chromoelectric string between the static quarks. Thus, at short distances (or high virtualities q^2) one could expect the decomposition of condensates, which means the scale of confinement (or the string tension) should disappear from the physical quantities at large q^2. In contrast, the nonperturbative scale can contribute as the factor in front of power corrections $1/q^2$ even at $q^2\to\infty$. So, in [19] several indications of linear term contribution at small distances were considered. We repeat the items relevant to the question on the static potential here.

First, the lattice simulation [20] does not show any change in the slope of the full $Q\bar{Q}$ potential as the distances are changed from the largest to the smallest ones where the Cou-

[3]The mass term generated by the sources should be equal to $\mathcal{L}\sim A^a_\nu A^b_\nu[\bar{n}T^aT^bn+\bar{n}T^an\cdot\bar{n}T^bn]$, so that the averaging of sources yields zero, if we suppose Eq. (18).

[4]This assumption corresponds to the definition of vacuum properties in QCD in terms of notations under the consideration, which is in agreement with the value of gluon condensate and Regge trajectories slope.

[5]The ambiguity in the choice of n_l can change the appropriate value of gluon condensate.

lombic part becomes dominant. Hence, no rapid energy jump, characteristic of the phase transition, is found on the lattice. An explicit subtraction of the perturbative corrections at small distances from the potential in the lattice gluodynamics was performed in [21]. This procedure gives an essential nonzero linear term at very small distances.

Second, there are the lattice measurements [22] of the fine splitting in the heavy quarkonium levels as a function of the heavy quark mass. The approach by Voloshin and Leutwyler [16] predicts a particular pattern of such a dependence. Indeed, the multipole expansion of heavy quarkonium interaction with the external gluon field leads to the dominant contribution by the second order of the chromoelectric dipole. Therefore, the quark distance squared appears as the leading term in the perturbation due to soft gluons at short distances. These predictions are very different from the evaluations based on the static quark potential with the linear term, such as the potential by Buchmüller and Tye [6]. The numerical results from the lattice simulations favor the linear correction to the potential at short distances.

Third, an interesting manifestation of short strings might be the power corrections to current correlation functions $\Pi_j(q^2)$. Calculations of a relevant coefficient in front of the $1/q^2$ terms involve the model assumptions. So, in [23] it was suggested that we simulate this power correction by a tachyonic gluon mass. The tachyonic mass can imitate the stringy piece in the potential at short distances. Rather unexpectedly, the use of the tachyonic gluon mass ($m_g^2=-0.5\mathrm{GeV}^2$) explains well the behavior of $\Pi_j(q^2)$ in various channels. This fact again implies that we see the confirmation of the short distance linear term in the potential.

Thus, we do not involve any additional assumptions on the possible scale and properties of quark-gluon condensate decomposition at short distances in the description of static potential in QCD.

C. Unified β function and potential

Buchmüller and Tye supposed the following procedure for the reconstruction of β function in the whole region of charge variation by the known limits of asymptotic freedom to a given order in α_s and confinement regime. So, in the framework of asymptotic perturbative theory (PT) to one loop, the $\beta_{\rm PT}$ is transformed to the Richardson one

$$\frac{1}{\beta_{\rm PT}(\mathfrak{a})}=-\frac{1}{\beta_0\mathfrak{a}^2}\Rightarrow\frac{1}{\beta_{\rm Rich}(\mathfrak{a})}=-\frac{1}{\beta_0\mathfrak{a}^2\left(1-\exp\left[-\frac{1}{\beta_0\mathfrak{a}}\right]\right)}. \tag{21}$$

The Richardson function has the essential peculiarity at $\mathfrak{a}\to 0$, so that the expansion is the asymptotic series in $\mathfrak{a}$. At $\mathfrak{a}\to\infty$ the β function tends to the confinement limit represented in Eq. (20).

According to the two-loop accuracy, following in the same way results in the β function by Buchmüller-Tye

$$\frac{1}{\beta_{\rm PT}(\mathfrak{a})}=-\frac{1}{\beta_0\mathfrak{a}^2}+\frac{\beta_1}{\beta_0^2\mathfrak{a}}\Rightarrow\frac{1}{\beta_{\rm BT}(\mathfrak{a})}=-\frac{1}{\beta_0\mathfrak{a}^2\left(1-\exp\left[-\frac{1}{\beta_0\mathfrak{a}}\right]\right)}+\frac{\beta_1}{\beta_0^2\mathfrak{a}}\exp[-l\mathfrak{a}]. \tag{22}$$

The exponential factor in the second term contributes to the next order in $\mathfrak{a}$ at small $\mathfrak{a}$, so that the perturbative limit is restored. However, we can easily find that the third coefficient of $\beta_{\rm BT}$ function is equal to

$$\beta_{2,\rm BT}=\frac{\beta_1}{\beta_0}(\beta_1-l\beta_0),$$

and it is negative at the chosen value of $l=24$ [6], which is in contradiction with the recent result [7,8], shown in Eq. (10).

To incorporate the three-loop results into the perturbative β function, we introduce

$$\frac{1}{\beta_{\rm PT}(\mathfrak{a})}=-\frac{1}{\beta_0\mathfrak{a}^2}+\frac{\beta_1+\left(\beta_2^V-\frac{\beta_1^2}{\beta_0}\right)\mathfrak{a}}{\beta_0^2\mathfrak{a}}\Rightarrow\frac{1}{\beta(\mathfrak{a})}=-\frac{1}{\beta_0\mathfrak{a}^2\left(1-\exp\left[-\frac{1}{\beta_0\mathfrak{a}}\right]\right)}+\frac{\beta_1+\left(\beta_2^V-\frac{\beta_1^2}{\beta_0}\right)\mathfrak{a}}{\beta_0^2\mathfrak{a}}\exp\left[-\frac{l^2\mathfrak{a}^2}{2}\right], \tag{23}$$

where again the exponential factor in the second term contributes to the next order in $\mathfrak{a}\to 0$. In the perturbative limit the usual solution

$$\mathfrak{a}(\mu^2)=\frac{1}{\beta_0\ln\frac{\mu^2}{\Lambda^2}}\left[1-\frac{\beta_1}{\beta_0^2}\frac{1}{\ln\frac{\mu^2}{\Lambda^2}}\ln\ln\frac{\mu^2}{\Lambda^2}+\frac{\beta_1^2}{\beta_0^4}\frac{1}{\ln^2\frac{\mu^2}{\Lambda^2}}\left(\ln^2\ln\frac{\mu^2}{\Lambda^2}-\ln\ln\frac{\mu^2}{\Lambda^2}-1+\frac{\beta_2^V\beta_0}{\beta_1^2}\right)\right] \tag{24}$$

is valid. Using the asymptotic limits of Eqs. (20) and (24), one can get the equations for any β function, satisfying these boundary conditions, as follows:

$$\ln\frac{\mu^2}{\Lambda^2}=\frac{1}{\beta_0\mathfrak{a}(\mu^2)}+\frac{\beta_1}{\beta_0^2}\ln\beta_0\mathfrak{a}(\mu^2)+\int_0^{\mathfrak{a}(\mu^2)}dx\left[\frac{1}{\beta_0 x^2}-\frac{\beta_1}{\beta_0^2 x}+\frac{1}{\beta(x)}\right], \quad (25)$$

$$\ln\frac{K}{\mu^2}=\ln\mathfrak{a}(\mu^2)+\int_{\mathfrak{a}(\mu^2)}^{\infty}dx\left[\frac{1}{x}+\frac{1}{\beta(x)}\right]. \quad (26)$$

In general, at a given β function, Eqs. (25) and (26) determine the connection between the scale Λ and the parameter of linear potential K,

$$k=2\pi C_F K.$$

Supposing Eq. (23), we can easily integrate out Eq. (25) to get the implicit solution of charge dependence on the scale

$$\ln\frac{\mu^2}{\Lambda^2}=\ln\left[\exp\left(\frac{1}{\beta_0\mathfrak{a}(\mu^2)}\right)-1\right]+\frac{\beta_1}{\beta_0^2}\left[\ln\frac{\beta_0\sqrt{2}}{l}-\frac{1}{2}\left(\gamma_E+E_1\left[\frac{l^2\mathfrak{a}^2(\mu^2)}{2}\right]\right)\right]+\frac{\beta_2^V\beta_0-\beta_1^2}{\beta_0^3}\frac{\sqrt{\frac{\pi}{2}}}{l}\mathrm{Erf}\left[\frac{l\mathfrak{a}(\mu^2)}{\sqrt{2}}\right], \quad (27)$$

where $E_1[x]=\int_x^\infty dt t^{-1}\exp[-t]$ is the exponential integral, and $\mathrm{Erf}[x]=(2/\sqrt{\pi})\int_0^x dt\exp[-t^2]$ is the error function.

Equation (27) can be inverted by the iteration procedure as it was explored in the derivation of Eq. (24). So, the approximate solution of Eq. (27) has the following form:

$$\mathfrak{a}(\mu^2)=\frac{1}{\beta_0\ln\left(1+\eta(\mu^2)\frac{\mu^2}{\Lambda^2}\right)}, \quad (28)$$

where

$$\eta(\mu^2)=\left(\frac{l}{\beta_0\sqrt{2}}\right)^{\beta_1/\beta_0^2}\exp\left[\frac{\beta_1}{2\beta_0^2}\left(\gamma_E+E_1\left[\frac{l^2\mathfrak{a}_1^2(\mu^2)}{2}\right]\right)-\frac{\beta_2^V\beta_0-\beta_1^2}{\beta_0^3}\frac{\sqrt{\frac{\pi}{2}}}{l}\mathrm{Erf}\left[\frac{l\mathfrak{a}_1(\mu^2)}{\sqrt{2}}\right]\right], \quad (29)$$

while $\mathfrak{a}_1$ is obtained in two iterations

$$\mathfrak{a}_1(\mu^2)=\frac{1}{\beta_0\ln\left(1+\eta_1(\mu^2)\frac{\mu^2}{\Lambda^2}\right)}, \quad (30)$$

$$\eta_1(\mu^2)=\left(\frac{l}{\beta_0\sqrt{2}}\right)^{\beta_1/\beta_0^2}\exp\left[\frac{\beta_1}{2\beta_0^2}\left(\gamma_E+E_1\left[\frac{l^2\mathfrak{a}_0^2(\mu^2)}{2}\right]\right)\right], \quad (31)$$

$$\mathfrak{a}_0(\mu^2)=\frac{1}{\beta_0\ln\left(1+\frac{\mu^2}{\Lambda^2}\right)}. \quad (32)$$

Taking the limit of $\mu^2\to 0$ we find the relation

$$\ln 4\pi^2 C_F\alpha'_P\Lambda^2=\ln\beta_0+\frac{\beta_1}{2\beta_0^2}\left(\gamma_E+\frac{l^2}{2\beta_0^2}\right)-\frac{\beta_2^V\beta_0-\beta_1^2}{\beta_0^3}\frac{\sqrt{\frac{\pi}{2}}}{l}, \quad (33)$$

which completely fixes the β function and charge in terms of scale Λ and the slope α'_P, since we have expressed the parameter l in terms of the above quantities.

Remember, that at $\mu^2\to\infty$ the perturbative expression Eq. (24) becomes valid as the limit of effective charge Eq. (28).

At the moment we are ready to discuss the numerical values of parameters.

D. Setting the scales

As we have already mentioned the slope of Regge trajectories, determining the linear part of the potential, is fixed as

$$\alpha'_P=1.04\ \ \mathrm{GeV}^{-2}.$$

We also use the measured value of QCD coupling constant [9] and pose

$$\alpha_s^{\overline{\mathrm{MS}}}(m_Z^2)=0.123,$$

as the basic input of the potential.

At the given choice of normalization value for the QCD coupling constant we get the scale $\Lambda_{n_f=5}^{\overline{\mathrm{MS}}}\approx 273$ MeV, which certainly differs from the world average value resulting in the analysis of the Particle Data Group [9], where $\Lambda_{n_f=5}^{\overline{\mathrm{MS}}}\approx 208^{+25}_{-23}$ MeV, which corresponds to the coupling constant $\alpha_s^{\overline{\mathrm{MS}}}(m_Z^2)=0.1181\pm 0.002$ [9]. However, this average value including various data is generally determined by the most precise measurements: the data on the hadronic events in the peak of Z boson at the CERN e^+e^- collider (LEP) (the hadronic width), the decays of τ lepton, the data on the deep inelastic scattering (DIS) for leptons off nucleons, and the lattice simulations for the systems of heavy quarkonia. In this set of estimates, the high energy measurements at LEP for Z and at the DESY ep collider HERA for the evolution of nucleon structure functions give the average values $\alpha_s^{\overline{\mathrm{MS}}}(m_Z^2)=0.123\pm 0.004$ and $\alpha_s^{\overline{\mathrm{MS}}}(m_Z^2)=0.122\pm 0.004$, respectively, while the evolution of structure functions at low virtualities, where an ambiguity in the description of nonperturbative effects and contributions of higher twists are essential, as well as the energy-dependent sum rules for the structure functions at low energies, significantly displace the common average value for the coupling constant extracted from the DIS data. Thus, we argue that the methodical uncertainty for such averaging is underestimated, since the low-energy data have some additional sources of theoretical uncertainties. The analysis of data on the decays of τ lepton resulting in $\alpha_s^{\overline{\rm MS}}(m_Z^2)=0.121\pm0.003$ is based on the sum rules, where the control of nonperturbative corrections is much better than in DIS, though there are some theoretical problems on the formulation of sum rules in the region of physical states in contrast with the classic variant of sum rules in the deep euclidean region. Finally, the lattice simulations investigate the splitting between the states of heavy quarkonia, i.e., they operate with the low-energy data and rely on an approximation with the zero number of light quarks $n_f=0$ or $n_f=2$ under the extrapolation to both the real number of $n_f=3$ and the region of high virtualities due to the evolution. A high accuracy of such lattice estimates is announced. As we have seen the spectroscopic characteristics for the systems of heavy quarks need an extremely careful interpretation, since the evolution of potential parameters from the region of bound states to the high virtualities is affected by the nonperturbative factors. Thus, we see that the normalization value of the QCD coupling constant accepted above agrees with the direct high-energy measurements, while the data obtained at low energies allow the agreement, if we take into account their systematic uncertainties, which are not well estimated.

Note that the decrease of normalization value to $\alpha_s^{\overline{\rm MS}}(m_Z^2)=0.120$, for example, leads to a discrepancy with the data on the splitting of heavy quarkonium masses between the levels of $1S$ and $2S$ states, which is very sensitive to the normalization of QCD coupling constant, so that instead of $M(2S)-M(1S)\approx 580$ MeV we get the value which is less by about 100 MeV. In this respect, the variation of the other dimensional parameter, the Regge trajectory slope, from the accepted value of $\alpha'_P=1.04\ \text{GeV}^{-2}$ to $\alpha'_P=0.87\ \text{GeV}^{-2}$ leads to an unessential change in both the splitting and the corresponding value for the scale in the coupling constant evolved to low virtualities.

Then, we evaluate

$$\alpha_V(m_Z^2)\approx 0.1306,$$

and put it as the normalization point for $\mathfrak{a}(m_Z^2)=\alpha_V(m_Z^2)/(4\pi)$. Further, we find the following values of Λ for the effective charge, depending on the number of active flavors:

$$\Lambda_{n_f=3}=643.48\ \text{MeV},\quad l=56, \tag{34}$$

$$\Lambda_{n_f=4}=495.24\ \text{MeV},\quad l=37.876, \tag{35}$$

$$\Lambda_{n_f=5}=369.99\ \text{MeV},\quad l=23.8967, \tag{36}$$

where we set the threshold values for switching the number of flavors to be equal to $m_5=4.6$ GeV and $m_4=1.5$ GeV.

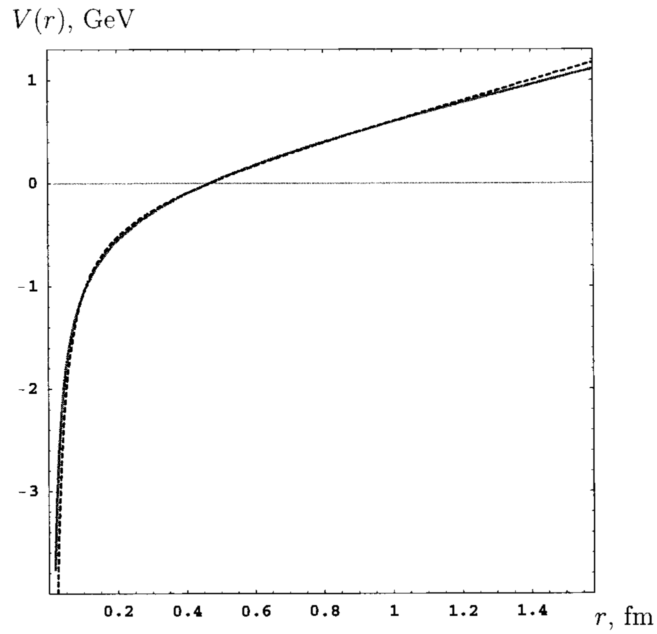

FIG. 1. The potential of static heavy quarks in QCD (solid line) in comparison with the Cornell model (dashed line) (up to an additive shift of energy scale).

After fixing the momentum space dependence of the charge, we perform the Fourier transform to get

$$V(r)=k\cdot r-\frac{8C_F}{r}u(r), \tag{37}$$

with

$$u(r)=\int_0^\infty \frac{dq}{q}\left(\mathfrak{a}(q^2)-\frac{K}{q^2}\right)\sin(q\cdot r),$$

which is calculated numerically at $r>0.01$ fm and represented in the MATHEMATICA file in the format of notebook at the site http://www.ihep.su/~kiselev/Potential.nb

Note, that at short distances the potential behavior is purely perturbative, so that at $r<0.01$ fm we put

$$V(r)=-C_F\frac{\overline{\alpha}_V(1/r^2)}{r}, \tag{38}$$

where the running $\overline{\alpha}_V(1/r^2)$ is given by Eq. (24) with the appropriate value of $\overline{\beta}_2^V$ at $n_f=5$, and with the matching potential Eq. (37) at $r_s=0.01$ fm, where we have found

$$\overline{\alpha}_V(1/r_s^2)=0.222\,13,$$

which implies $\Lambda_{n_f=5}^{\overline{V}}=617.42$ MeV.

Thus, we have completely determined the model for the potential of static heavy quarks in QCD. In Fig. 1 we present it versus the distance between the quarks. As we can see the potential is very close to what was obtained in the Cornell model in the phenomenological manner by fitting the mass spectra of heavy quarkonia.

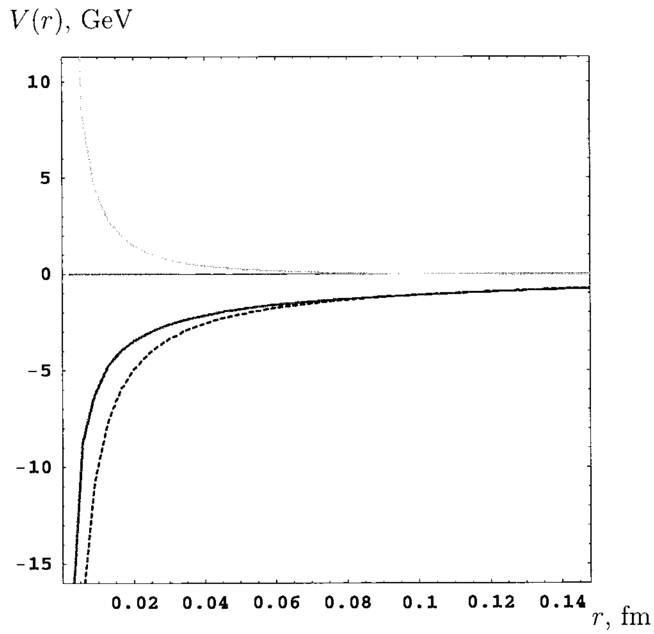

FIG. 2. The potential of static heavy quarks in QCD (solid line) in comparison with the Cornell model (dashed line, up to an additive shift of energy scale) and the difference between them (upper curve) at short distances as caused by the running of coupling in QCD.

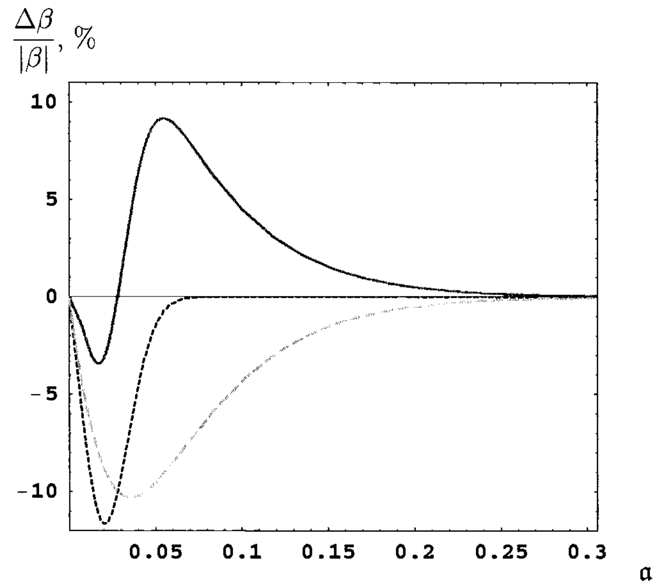

FIG. 3. The differences between the β functions vs the effective charge. The value of $(\beta-\beta_{\rm BT})/|\beta|$ is shown by the solid line, $(\beta-\beta_{\rm Rich})/|\beta|$ is given by the short-dashed line, and $(\beta_{\rm BT}-\beta_{\rm Rich})/|\beta_{\rm BT}|$ is represented by the long-dashed line.

The visual deviation between the QCD potential derived and the Cornell model at long distances is caused by a numerical difference in the choice of string tension: we adopt the value given by Buchmüller and Tye, while in the Cornell model the tension is slightly greater than that we have used. A more essential point is the deviation between the potentials at short distances (see Fig. 2), because of the running of the coupling constant in QCD in contrast to the constant effective value in the Cornell model.

For comparison, we show the differences between the β functions Eqs. (21), (22), and (23) in Fig. 3 at the fixed values of l and $n_f=3$. Wee see that the asymptotic perturbative expansion of β at $\mathfrak{a}\to 0$ dominates at $\mathfrak{a}<\mathfrak{a}_0$, where $\mathfrak{a}_0 \approx 0.03$ corresponding to $\alpha_{V,0}\approx 0.37$. This value of coupling $\alpha_{V,0}$ coincides with the effective Coulomb constant used in the Cornell model. At larger values of coupling the contributions related with the confinement regime are essential.

Two comments are to the point. First, the resulting potential is obtained by the perturbative normalization to the measured value of $\alpha_s^{\overline{\rm MS}}(m_Z^2)$ as combined with the three-loop evolution to the lower virtualities. Second, the running of the coupling constant is modified (numerically the deviation from the perturbative regime begins at $\mu<3-4$ GeV) to reach the confinement limit at $\mu\to 0$, so that the perturbative connection between the scales Λ and $\Lambda^{\overline{\rm MS}}$ is broken at virtualities under touch by the charmed and bottom quarks, which was the reason for the error in the assignment of $\Lambda^{\overline{\rm MS}}$ by Buchmüller and Tye.

III. HEAVY QUARK MASSES AND LEPTONIC CONSTANTS

Considering the characteristics of heavy quark bound states we should emphasize a significant necessity to separate two distinct theoretical problems. The first problem is calculation of the heavy quark potential, where the leading approximation is the static limit of $m_Q\to\infty$ in the operator product expansion over the powers of inverse heavy quark mass. We have considered this problem in Sec. II. The other problem is the calculation of bound state masses. In the heavy quarkonium the kinetic energy of quark motion is comparable with the potential energy. So, the leading approximation for the effective Lagrangian in the operator product expansion over the inverse heavy quark mass is the sum of the nonrelativistic kinetic term and the static potential, which give the dominant contribution in the Schrödinger equation for the bound states. Corrections are relativistic

TABLE I. The masses of charmonium as predicted in the present paper ($\rm K^2O$) in comparison with the experimental data elaborated as described in the text.

State (nL)	M ($\rm K^2O$)	$\bar{M}$ (exp.)	State (nL)	M ($\rm K^2O$)	$\bar{M}$ (exp.)
$1S$	3.068	3.068	$2P$	3.493	3.525
$2S$	3.670	3.671	$3P$	3.941	
$3S$	4.092	4.040	$3D$	3.785	3.770

TABLE II. The masses of bottomonium as predicted in the present paper (K^2O) in comparison with the experimental data elaborated as described in the text.

State (nL)	M, (K^2O)	$\overline{M}$ (exp.)	State (nL)	M (K^2O)	$\overline{M}$ (exp.)
$1S$	9.446	9.446	$2P$	9.879	9.900
$2S$	10.004	10.013	$3P$	10.239	10.260
$3S$	10.340	10.348	$3D$	10.132	
$4S$	10.606	10.575	$5S$	10.835	10.865

terms in the kinetic energy and perturbations of the static potential in the form of operators suppressed by the inverse powers of heavy quark mass, as well as nonpotential retardation effects. The magnitude of such corrections can be restricted numerically, which leads to a systematic uncertainty in the calculations of mass spectra for the heavy quarkonia in the framework of the potential approach with the static potential.

A. Masses

The determination of potential provides us with the extraction of heavy quark masses in the static approximation by comparison of heavy quarkonium mass spectra with the calculated ones. The predicted charmonium and bottomonium masses are presented in Tables I and II[6] at the following values of heavy quark masses in the potential approach:

$$m_c^V = 1.468 \text{ GeV}, \quad m_b^V = 4.873 \text{ GeV}, \tag{39}$$

without taking into account relativistic corrections, which can be sizable for the charmonium (say, $\Delta M(\bar{c}c) \sim 40$ MeV). At the moment, the only measured splitting of nS levels is that of η_c and J/ψ, which allows us to evaluate the so-called spin-averaged mass

$$\overline{M}(1S) = (3M_{J/\psi} + M_{\eta_c})/4.$$

Supposing the simple relation [4], $\overline{M}(ns) = M_V(nS) - (1/4n)(M_{J/\psi} - M_{\eta_c})$, we also estimate the expected values for the excited states with an accuracy better than 10 MeV (we believe). For the P-wave levels we explore the masses

$$\overline{M}(P) = M_1 + \tfrac{1}{3}(M_2 - M_0) + \tfrac{2}{9}(M_2 - M_1 + 2(M_0 - M_1)),$$

where M_J denotes the mass of state with the total spin J and the sum of quark spins $S=1$, and we have supposed the spin-dependent forces in the form

$$V_{SD} = A(\mathbf{L}\cdot\mathbf{S}) + B(\mathbf{L}\cdot\mathbf{S})^2 - \tfrac{1}{3}B\mathbf{L}^2\cdot\mathbf{S}^2,$$

TABLE III. The masses of $\bar{b}c$ as predicted in the present paper (K^2O) in comparison with the experimental data.

State (nL)	M (K^2O)	$\overline{M}$ (exp.)	State (nL)	M (K^2O)	$\overline{M}$ (exp.)
$1S$	6.322	6.40	$2P$	6.739	
$2S$	6.895		$3P$	7.148	
$3S$	7.279		$3D$	7.013	

where the third term corresponds to the third term in the above expression for $\overline{M}(P)$ and it results in the L-dependent shift of levels.

We have also supposed

$$M_\Upsilon - M_{\eta_b} \approx \frac{\alpha_s(m_b)}{\alpha_s(m_c)} \frac{m_c^2}{m_b^2} \frac{|R_{\bar{b}b}(0)|^2}{|R_{\bar{c}c}(0)|^2} (M_{J/\psi} - M_{\eta_c}) \approx 56 \text{ MeV}.$$

We have found that the sizes of quarkonia are the same as they were predicted by Buchmüller and Tye, while the masses of states are slightly different since we have used the other prescription for the input values of ground state masses:

$$M_{\bar{c}c}(1S) = 3.068 \text{ GeV}, \quad M_{\bar{b}b}(1S) = 9.446 \text{ GeV}.$$

Then, we predict the masses of $\bar{b}c$ quarkonium,[7] as shown in Table III. The calculated values of masses agree with those estimated in the Buchmüller-Tye and Martin potentials [25]. The wave functions at the origin are related with the production rates of heavy quarkonia. These parameters are close to what was predicted in the BT potential, but slightly smaller because of both the change in the charmed quark mass and the asymptotic behavior at $r \to 0$.

At the moment we have fixed the potential masses of heavy quarks Eq. (39) as independent of scale. To compare with the masses evaluated in the framework of QCD sum rules, we note that in the sum rules for the heavy quarkonia one usually explores the NRQCD [26] with the perturbative potential Eq. (14) explicitly dependent of the normalization point μ (referred as μ_{soft} in the SR). We have determined that at short distances and high μ_{soft} the perturbative potential Eq. (14) and that of the present paper coincide with each other, while a deviation appears at $r \gg 1/\mu_{\text{soft}}$. However, at the distances characteristic for the ground states of heavy quarkonia, $\langle r_{\bar{b}b(1S)} \rangle \approx 0.22$ fm and $\langle r_{\bar{c}c(1S)} \rangle \approx 0.42$ fm, the shape of the potential can be approximated by the perturbative term at $\mu_{\text{soft}} = 1.5$–2.0 GeV (see Figs. 4 and 5) with the additive shift of energy scale $\delta V(\mu_{\text{soft}})$, which is defined by the expression

$$\delta V(\mu_{\text{soft}}) = [V(r) - V_{\text{pert}}(r;\mu_{\text{soft}})]\big|_{r=\frac{1}{\mu_{\text{soft}}}\zeta}, \tag{40}$$

[6]We suppose that the $\psi(3770)$ state is a mixture of $3S$ and $3D$ levels with unessential shift of $3D$ mass.

[7]The experimental error in the ground state mass is still large, $\delta M = \pm 0.39$ GeV [24].

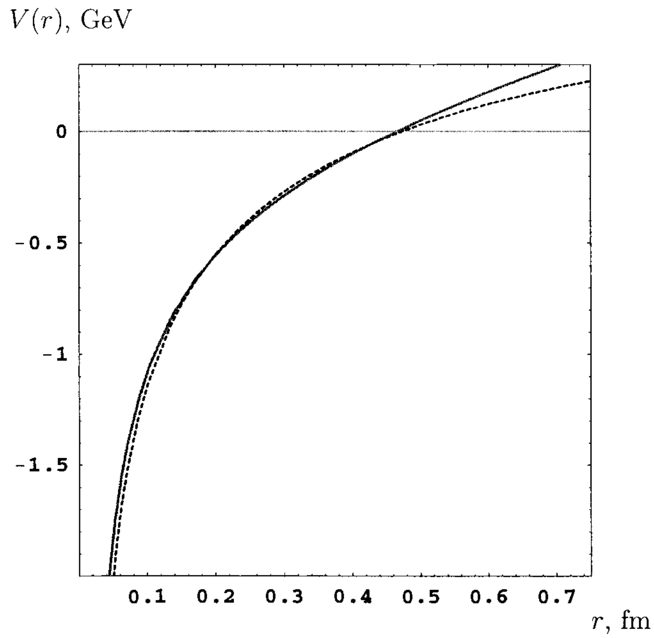

FIG. 4. The potential of static heavy quarks in QCD (solid line) in comparison with the perturbative term Eq. (14) at $\mu=1.5$ GeV (dashed line) (up to an additive shift of energy scale).

where the parameter ζ has been put in the region of $\zeta=1-2$, where the energy shift δV has varied slightly by about 30–40 MeV, which is, on the first hand, a characteristic uncertainty of potential approach, and on the other hand, it points to a similar form of perturbative potential with the calculated model potential in the region of distance variation. The dependence of energy shift is represented in Fig. 6.

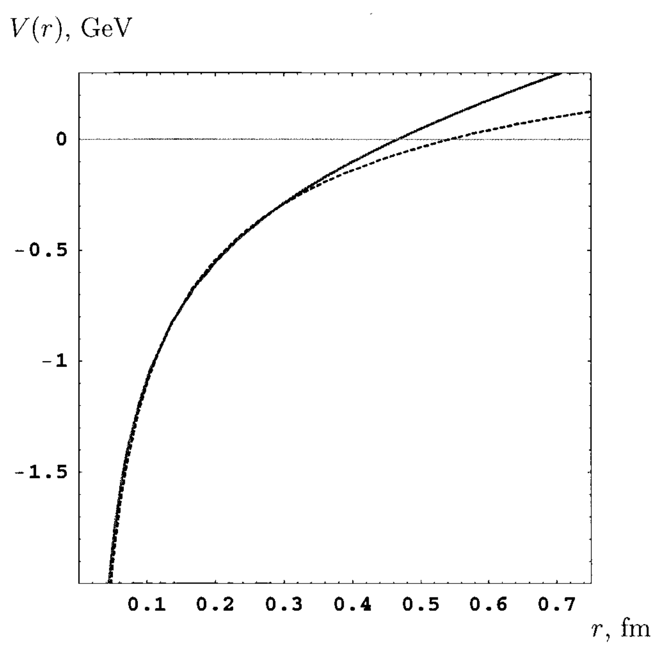

FIG. 5. The potential of static heavy quarks in QCD (solid line) in comparison with the perturbative term Eq. (14) at $\mu=2.0$ GeV (dashed line) (up to an additive shift of energy scale).

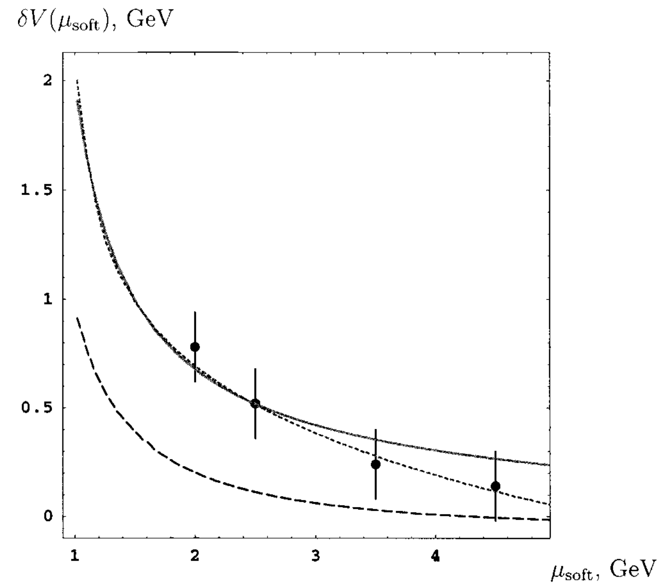

FIG. 6. The value of additive shift of energy scale to match the perturbative μ-dependent potential with that of calculated in QCD. The solid and dashed lines correspond to the two- and one-loop matching. The points give the result of sum rules for bottomonium in comparison with the dotted curve following from the relation between the running and pole masses at scale μ.

So, if we redefine the heavy quark masses[8] by

$$m^{\rm pole}(\mu)_{b,c}=m^V_{b,c}+\tfrac{1}{2}\delta V(\mu),$$

the solution of Schrödinger equation with the perturbative potential and $m^{\rm pole}(\mu)$ results in the quarkonia masses close to the experimental values. Thus, we have matched the values of potential masses m^V in the QCD potential with the perturbative pole masses standing in the two-loop calculations. We stress that the dependence on the soft scale in both the energy shift $\delta V(\mu)$ and the pole mass $m^{\rm pole}(\mu)$ does not reflect a nonzero anomalous dimension, since these quantities are renormalization group invariants. This scale dependence is due to the truncation of perturbative expansion, wherein the coefficients in front of powers of coupling constant can contain the factorial growth (the renormalon), so that even at the scale close to the charmed quark mass the infrared singularity in the running coupling constant of QCD provides the significant custodial scale dependence.

[8]This redefinition is the indication of perturbative renormalon (see review in [27]). Indeed, there are two sources for the deviation δV. The first is the linear confining term in the potential of static quarks. However, it is a small fraction of δV. The second source is the infrared singularity in the perturbative running coupling. One can easily find that subtracting the singular term of the form $\sim 1/(\mu_{\rm soft}-\Lambda)$ from δV results in a small value slowly depending on $\mu_{\rm soft}$. In the effective theory for the nonrelativistic heavy quarks, the subtraction that connects the pole mass and the threshold mass can be calculated explicitly (see [28] and references therein).

Numerically, we estimate the running masses $\bar{m}(\bar{m})$ in the $\overline{\rm MS}$ scheme using the two- and three-loop relations with the pole mass derived in [29,30] and adjusting the scale $\mu_{\rm soft}$ to be equal to $\bar{m}$. So, in two loops [29] we get

$$\bar{m}_c(\bar{m}_c)_{2\ \rm loops} = 1.40 \pm 0.09\ \ \text{GeV},$$

$$\bar{m}_b(\bar{m}_b)_{2\ \rm loops} = 4.20 \pm 0.06\ \ \text{GeV},$$

while the three-loop approximation [30], which is consistent with the three-loop evolution of coupling constant, results in slightly smaller masses, especially for the charmed quark, where the uncertainty of estimate increases because of stronger sensitivity of quantities involved to the scale variation

$$\bar{m}_c(\bar{m}_c)_{3\ \rm loops} = 1.17 \pm 0.10\ \ \text{GeV},$$

$$\bar{m}_b(\bar{m}_b)_{3\ \rm loops} = 4.15 \pm 0.06\ \ \text{GeV},$$

which are in agreement with the various estimates in the sum rules on m_b [31–34] and m_c [35].[9]

The uncertainty of estimates is determined by the deviations in the calculations of heavy quarkonium masses $\frac{1}{2}\delta M = 20$ MeV (as shown in Tables I and II) and the error in the extraction of δV mentioned above. The uncertainty in the running mass of a charmed quark is slightly larger than in the bottom mass since, in addition, its value is more sensitive to a small variation of scale, pole mass and energy shift.

Note, that the calculations in the framework of sum rules were performed for the b-quark mass in both the full QCD [36] and the effective theory of nonrelativistic heavy quarks NRQCD [31–34,37]. The mass extraction of Ref. [37] has been carried out in the nonrelativistic effective theory at next-to-leading order (NLO), whereas Refs. [31–34] carried out next-to-NLO (NNLO) analyses in the same framework. The calculations in the nonrelativistic effective theory are the calculations in the framework of first principles in QCD, where the results of full QCD are determined in a systematic expansion in α_s and the velocity. In Ref. [36] the analysis has also been carried out in the nonrelativistic situation, but no systematic expansion in α_s and the velocity has been carried out. That the results for the $\overline{\rm MS}$ mass obtained in [36] agree with the other analyses is not understood and requires further examination (see the conclusions of [38]).

Recently, the charmed quark mass was evaluated from the NRQCD sum rules in [39], so that the result on the running mass is in a good agreement with the value given above, too. There is a recent sum rule extraction [40] of the bottom $\overline{\rm MS}$ mass, where the charmed quark mass effects are also included. The estimate of potential approach under consideration is in good agreement with this recent SR result.

In [33] the dependence of "pole" mass on the scale $\mu_{\rm soft}$ was explicitly calculated in the NNLO. The uncertainty of mass extraction from the sum rules for bottomonium was given by 0.1 GeV for the running $\overline{\rm MS}$ mass and 0.06 GeV for the low-energy running mass ("kinetic" mass). The result on the b-quark pole mass depends on both the scale of calculations and the order in α_s of perturbative QCD. To compare the results in the sum rules with those given in the present paper we fix the order in α_s by the two-loop corrections. Then we have found that, say, at $\mu_{\rm soft} = 2.5$ GeV the results of estimates in the perturbative potential approach and in the framework of sum rules are the same within the uncertainty mentioned. So, putting the above value as the matching point we show the sum rule results in the form of energy shift in Fig. 6. For the sake of representability in Fig. 6 we show the μ-dependent "pole" mass extracted in [33] with the uncertainty of $\delta m = 80$ MeV, which is a characteristic inherent error for the short-distance masses in the analysis of [33]. Despite the various choices for the normalization of QCD coupling constant (in [33] $\alpha_s^{\overline{\rm MS}}(m_Z^2) = 0.118$), we see a good agreement between the μ dependencies of both the energy shift in the perturbative potential with respect to the static potential of QCD and the variation of perturbative "pole" mass of b quark in the sum rules of QCD. As for the one-loop matching of the perturbative potential, we mention only that the corresponding sum rules in the NLO give the value of energy shift close to zero at $\mu_{\rm soft} > 2$ GeV within the uncertainty of the method, and this estimate is consistent with the result of potential approach as shown in Fig. 6. Thus, the energy shift of perturbative potential with the two-loop matching of V and $\overline{\rm MS}$ schemes indicates the form of QCD potential in agreement with the corresponding soft scale dependence of perturbative pole mass in sum rules of QCD for the bottomonium.

At the moment we can compare the obtained μ dependence of "pole" mass with the relation between the running $\overline{\rm MS}$ mass of heavy quark and the pole mass derived in [41], where we find

$$m^{\rm pole} = \bar{m}(\mu)\left(1 + c_1(\mu)\frac{\alpha_s^{\overline{\rm MS}}(\mu^2)}{4\pi} + c_2(\mu)\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu^2)}{4\pi}\right)^2\right), \tag{41}$$

with

$$c_1(\mu) = C_F(4 + 3L), \tag{42}$$

$$\begin{aligned} c_2(\mu) = {} & C_F C_A\left(\frac{1111}{24} - 8\zeta(2) - 4I_3(1) + \frac{185}{6}L + \frac{11}{2}L^2\right) \\ & - C_F T_F n_f\left(\frac{71}{6} + 8\zeta(2) + \frac{26}{3}L + 2L^2\right) \\ & + C_F^2\left(\frac{121}{8} + 30\zeta(2) + 8I_3(1) + \frac{27}{2}L + \frac{9}{2}L^2\right) \\ & - 12 C_F T_F(1 - 2\zeta(2)), \end{aligned} \tag{43}$$

where $I_3(1) = \frac{3}{2}\zeta(3) - 6\zeta(2)\ln 2$, and $L = 2\ln(\mu/m^{\rm pole})$. At $\mu = m^{\rm pole}$, the result of [29] is reproduced. We check that the logs in the definitions of $c_{1,2}$ can be removed by the expression of running values $\bar{m}(\mu)$ and $\alpha_s^{\overline{\rm MS}}(\mu)$ in terms of

[9] Note, there is the difference between the usually quoted values of $\bar{m}(\bar{m})$ and $\bar{m}(m^{\rm pole})$.

$\overline{m}(m^{\rm pole})$ and $\alpha_s^{\overline{\rm MS}}(m^{\rm pole})$ in Eq. (41). Nevertheless, we find that the explicit μ dependence in Eq. (41) repeats the form of renormalon contribution as we see it in the perturbative potential, where a similar effect takes place because of both the truncation of perturbative series and the infrared pole in the running coupling constant of QCD. Following Eq. (41), we show the value of difference $2(m_b^{\rm pole}(\mu)-m_b)$ in Fig. 6 at $\overline{m}(\overline{m})=4.3$ GeV. We see that, first, the results of QCD sum rules in [33] agree with the values expected from Eq. (41), and second, the μ-dependent shift of pole mass approximately coincides with the shift of perturbative potential with respect to the static QCD potential free off renormalon ambiguity caused by infrared singularity of perturbative coupling constant at finite energy scale. This fact implies the cancellation of infrared uncertainties. Thus, we can define the unambiguous pole mass by

$$\hat{m}^{\rm pole}=m^{\rm pole}(\mu)-\tfrac{1}{2}\,\delta V(\mu), \qquad (44)$$

where we use the pole mass of Eq. (41). The basis for the validity of Eq. (44) was observed in [42], where in the context of perturbative bottom mass extractions, the cancellation of the leading renormalon at $u=1/2$ of the Borel plane in the total static perturbative energy of a heavy $Q\overline{Q}$ pair was shown.

We find that for the bottom quark the defined mass is given by the value of mass extracted from the potential approach

$$\hat{m}_b^{\rm pole}\approx m_b^V,$$

with the accuracy about 80 MeV.

B. Heavy quark masses and PNRQCD

In this section we discuss the modern development in the theory of heavy quarkonium $Q\overline{Q}'$ on the basis of effective theory called PNRQCD [10], naturally incorporating the potential interactions between the heavy quarks and external ultrasoft fields in QCD, and compare the PNRQCD results with the values of heavy quark masses obtained above in the QCD potential of static quarks.

First, PNRQCD argues that in the heavy quarkonium the nonrelativistic motion of heavy quarks inside the bound state allows us to introduce three actual physical scales: the heavy quark mass m, the soft scale of heavy quark momentum inside the hadron mv, and ultrasoft scale of energy mv^2, which are distinctly separated by a small parameter v being the velocity of heavy quark. After matching with full QCD at a hard scale $\mu_{\rm hard}\sim m$, in NRQCD the hard fields are integrated out, which results in the perturbative Wilson coefficients of OPE in the effective theory, and we deal with the heavy quarks interacting with the gluons at virtualities $\mu_{\rm fact,\,soft}$ about mv. In order to consider the heavy quark fields at lower μ up to mv^2 we should introduce the effective Lagrangian of PNRQCD, where the soft fields are integrated out, and we deal with the potential interaction of heavy quarks and the ultrasoft external gluon fields in the framework of multipole expansion. The matching of PNRQCD with NRQCD takes place at a scale $\mu_{\rm fact}\sim mv$. Recently, the effective theory of vNRQCD was formulated in [11], using the velocity renormalization group [43] to match the vNRQCD operators with the full QCD at a scale about m with the single-step evolution to a soft scale, which can be either mv or mv^2. The current status of vNRQCD provides us with the one-loop matching of heavy quark potential to order v^2, i.e., up to spin-dependent $1/m^2$ terms, which are beyond the current consideration. Therefore, we concentrate our discussion on PNRQCD.

The PNRQCD Lagrangian has the following form:

$$\begin{aligned}\mathcal{L}_{\rm PNRQCD}={\rm Tr}\Bigg\{ & S^\dagger\Bigg(i\partial_0-\frac{\mathbf{P}^2}{4m}-\frac{\mathbf{p}^2}{m}+\frac{\mathbf{p}^4}{4m^3}-V_s(r)-\frac{V_s^{(1)}}{m}\\ & -\frac{V_s^{(2)}}{m^2}+\dots\Bigg)S+O^\dagger\Bigg(iD_0-\frac{\mathbf{P}^2}{4m}-\frac{\mathbf{p}^2}{m}+\frac{\mathbf{p}^4}{4m^3}\\ & -V_0(r)-\frac{V_0^{(1)}}{m}-\frac{V_0^{(2)}}{m^2}+\dots\Bigg)O\Bigg\}\\ & +gV_A(r){\rm Tr}\{O^\dagger\mathbf{r}\cdot\mathbf{E}\,S+S^\dagger\mathbf{r}\cdot\mathbf{E}O\}\\ & +g\frac{V_B(r)}{2}{\rm Tr}\{O^\dagger\mathbf{r}\cdot\mathbf{E}O+O^\dagger O\mathbf{r}\cdot\mathbf{E}\}-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a},\end{aligned} \qquad (45)$$

where $\mathbf{P}$ is the momentum associated to the center-of-mass coordinate. In Eq. (45) the $1/m$ corrections to V_A, V_B, and to pure gluonic operators as well as the higher order terms in the multipole expansion are not displayed. To the leading order the singlet and octet operators S, O are represented by the appropriate products of nonrelativistic heavy quark and antiquark spinors. The matching of S and O operators with the NRQCD spinors was done in [10] up to three loops for both the potentials and the normalization factors in OPE. In this Lagrangian the singlet and octet potentials $V_s(r)$ and $V_0(r)$ are treated as the corresponding Wilson coefficients in front of bilinear forms in S and O to the leading order in $1/m$. In Ref. [10] the authors show that this definition of static quark potential is consistent with the definition in terms of the Wilson loop Eq. (1).

The other result of PNRQCD is the cancellation of renormalon ambiguity in the sum of heavy quark pole masses and the potential up to two loops, which is a confirmation of general consideration in QCD, that was first derived in [42].

A new feature appears by the consideration of three-loop leading log matching of V and $\overline{\rm MS}$ schemes. So, for the distance-dependent running coupling the result reads off

$$\alpha_V(1/r^2,\mu)=\alpha_{\overline{\rm MS}}(1/r^2)\Bigg\{1+(a_1+2\gamma_E\beta_0)\frac{\alpha_{\overline{\rm MS}}(1/r^2)}{4\pi}+\left[\gamma_E(4a_1\beta_0+2\beta_1)+\left(\frac{\pi^2}{3}+4\gamma_E^2\right)\beta_0^2+a_2\right]\frac{\alpha^2_{\overline{\rm MS}}(1/r^2)}{16\pi^2}+\frac{C_A^3}{12}\frac{\alpha^3_{\overline{\rm MS}}(1/r^2)}{\pi}\ln r\mu\Bigg\}, \quad (46)$$

where the two-loop contribution was taken from [7,8] and it coincides with Eq. (14), of course. However, the three-loop term leads to the explicit dependence on the scale in the perturbative PNRQCD calculations, which has to be expected from the general note on the infrared singularity observed by Appelquist, Dine, and Muzinich [12], that was rederived in PNRQCD by supplementing a certain infrared subtraction. This dependence was considered in [10] for two cases, when the scales of confinement $\Lambda_{\rm QCD}$ and binding energy mv^2 have the arrangements: (a) $\Lambda_{\rm QCD}\gg mv^2$ or (b) $mv^2\gg\Lambda_{\rm QCD}$. If (a), the singlet potential of static quarks suffers from the nonperturbative effects, and it can be treated only after introduction of some model dependent terms coming from the ultrasoft gluons, which form the gluon sea in the heavy quarkonium, so that the sea has its excitations, and the characteristic excitation energy of gluelumps should replace the scale μ, this results in the scale-independent nonperturbative potential.[10] If (b), the potential is purely perturbative. However, calculating the physical quantities such as the masses of bound states, we have to take into account the contributions coming from the perturbative ultrasoft gluons with the virtualities less than μ, which can produce a μ-dependent shift of energy, which should be cancelled with the μ dependence in the potential Eq. (46) and, probably, in the heavy quark masses. In both cases, the perturbative calculations of singlet potential[11] explicitly indicate the necessity of taking into account the gluon degrees of freedom inside the heavy quarkonium. As was noted in [10], apparently, this feature is characteristic for the non-Abelian theory [see the factor of C_A in front of the log term in Eq. (46)].

To our opinion, this dependence of potential on the ultrasoft gluon fields (the infrared singularity in terms of Appelquist, Dine and Muzinich) inside the heavy quarkonium naturally indicates the formation of gluon string between the heavy quarks at long distances. Indeed, expression (45) was derived under the following arrangement of scales: $r^{-1}\sim mv$, $mv^2<\mu<mv$. So, if we put

$$\mu=\frac{u}{r}-\sigma r,$$

with

$$v<u<1,$$

and

$$\sigma\ll\frac{u}{m^2v^2},$$

then, perturbatively expanding in the small parameter σr, we get the linear correction to the potential in PNRQCD, so that

$$\Delta V_{\rm PNRQCD}=\Delta k\cdot r,$$

where

$$\Delta k=\frac{C_FC_A^3}{12\pi}\alpha^4_{\overline{\rm MS}}\frac{\sigma}{u}\approx\frac{C_A^3}{12\pi}\alpha^3_{\overline{\rm MS}}\sigma,$$

so that we have dropped the scale dependence of the strong coupling constant, since it is beyond the accuracy under study, and we have substituted the Coulomb relation for the quark velocity inside the bound state $u\approx C_F\alpha_{\overline{\rm MS}}$ to the given order. Numerically, for the charmed quarks this perturbative contribution could be on the order of $\Delta k\sim 0.1$ GeV^2. Thus, we can motivate the relation between the nonperturbative string and the three-loop scale dependent term in the PNRQCD potential.

Indifferently of the arrangement for the confinement and binding energy scales, the introduction of such a string should remove the explicit dependence of full potential on the scale. This has been done above by introduction of the unified β function of coupling in the V scheme. This solution of the problem qualitatively agrees with the consideration in PNRQCD since, first, in the perturbative regime the contribution of the log term is negligibly small as we see for the linear confining term of the potential at short distances, and, second, at long distances the nonperturbative confining term is essential, where the string tension is the natural physical scale. In the static potential of QCD given above we do not consider possible "nontrivial" excitations with the broken string geometry, where the break point moves on the string with the speed of light. Such excitations would correspond to hybrid states with the gluelumps. Thus, we find that the QCD potential of static quarks in the form offered in the present paper has no conflicts with the current status of PNRQCD.

However, in our opinion the problem can be much deeper. The static potential, introduced by the Wilson loop, is renormalization group invariant, and it does not contain any separation between the potential gluons and the ultrasoft gluons forming the sea, since it gives the total energy of dynamical fields. In contrast, the PNRQCD introduced the singlet potential as the Wilson coefficient in front of the four quark operator, so that it intrinsically operates with the separation of potential and sea, as well as the nonrelativistic quarks, which act as sources, so that some gluons with virtualities greater than μ are considered as emitted, while others with virtualities less than μ are included in the origin of sources, and the gluons with virtualities about mv mediate the potential interaction. Generally, this separation of heavy quarks,

[10] Possible nonpotential terms are discussed in [10].

[11] We are not concerned about the octet potential of static quarks in the present consideration, although some qualitative conclusions could be straightforwardly generalized from the singlet state to the octet one.

potential gluons, and sea gluons in the operator product expansion can involve nonzero anomalous dimensions for the singlet PNRQCD-potential. This fact does not contradict the OPE basis, but it reflects the point that the static potential of the Wilson loop generally differs from the PNRQCD potential. In addition, the ultrasoft gluon sea introduced in PNRQCD in terms of multipole interaction with local external chromoelectric and chromomagnetic fields is not a local object, indeed.

A point should be considered on the linear confining term of potential. In [10] a model of infrared behavior was used, so that at long distances between the heavy quarks the ultrasoft correction was derived in the form of constant energy shift δV_0 and quadratic term $\sigma_2 r^2$. The corresponding conclusion was drawn to stress that the linear term could appear in a more complicated case of infrared behavior. We show in the previous sections how this confinement regime can be reached.

Recently, several papers [44,45] were devoted to the calculations of ground states in the heavy quarkonia in the way, combining the PNRQCD potential with the nonperturbative corrections to the binding energy as they were produced by the multipole expansion of QCD [16] in the form of PNRQCD explicitly shown in Eq. (45). Reference [44] does not strictly estimate the gluon condensate effects in the multipole expansion, and it presents purely perturbative results. It follows the perturbative ground state mass technique as a mass definition that leads to the cancellation of the $u=1/2$ renormalon that was considered in the approach of upsilon expansion introduced by Hoang *et al.* in [46]. So, in [44] the perturbative mass of the B_c meson was calculated on the base of perturbative expansion for the static potential with the leading approximation in the form of Coulomb wave functions. As we see above the perturbative potential suffers from the renormalon ambiguity. In order to remove this dependence on the choice of scale μ in the potential, the authors of [44] calculated the masses of J/ψ and Υ in the same technique at the same point μ and inverted the problem on the heavy quark masses by equalizing the perturbative masses of ground states in the charmonium and bottomonium to the measured values. This procedure leads to the μ-dependent pole masses of heavy quarks as expressed by the series in $\alpha_s(\mu)$. We expect that such a procedure could cancel the renormalon with an accuracy of about 50 MeV in the mass of hadron. As a results, the perturbative mass of B_c has quite a stable value

$$M_{\rm pert}(B_c)=6326^{+29}_{-9}\ \ \text{MeV}, \tag{47}$$

in the range of $1.2<\mu<2.0$ GeV, which should be compared with the results in Table III and the range of μ described above in the study of matching the perturbative potential with the full QCD potential. The authors of [44] did not present the μ-dependent heavy quark masses. Nevertheless, due to the almost coinciding estimates of B_c mass in Eq. (47) and Table III, we expect that this dependence should be given by the form of $\delta V(\mu)$.

In Ref. [45] the same technique for the perturbative contribution with the account for both the gluon condensate corrections in the multipole expansion of QCD and a small $\alpha_s^5\log\alpha_s$ term, was used to extract the heavy quark masses. The authors determined the "pole" mass, which is scale dependent, indeed, by putting $\mu=C_F\alpha_s m_Q$ in the potential. As we understand, they introduced the mass suffered from the renormalon and got

$$m_b=5022\pm 58\ \ \text{MeV},$$

which is greater than we determine in the current presentation. The reason is quite evident. It is the energy shift $\delta V(\mu)$. The running $\overline{\rm MS}$ mass quoted in [45] is about 260 MeV greater than we find in the same order in α_s for the relation between the pole and running masses. The difference becomes unessential by using the three-loop matching of the masses in [45], however, the same correction will also decrease the value obtained in the spectroscopy with the full QCD potential. Thus, to our opinion the values of heavy quark masses given in [45] should be kept with a large care.

Finally, in [47] the dependence of potential on the finite heavy quark masses was considered. This dependence is due to the smooth variation of the number of active flavors in the expressions for the coefficients of the perturbative β function as well as in the matching coefficients of α_V. As we have described above we use the step-like change of active flavor number, which infers implicit model dependence, which is practically unavoidable in the case under study.

As for the lattice simulations in QCD for the relevant problem, a review can be found in Ref. [48]. We emphasize only that the lattice potential of static quarks is close to what is given by the Cornell model. A modern review of phenomenological potential models can be found in the lectures [49]. The finite mass effects in the nonrelativistic bound states was recently considered at next-to-leading order in [50] and [51]. A next-to-next-to-leading order analysis of light quark mass effects in the heavy nonrelativistic $Q\bar{Q}$ systems was given in [40]. Some applications of PNRQCD to the heavy quarkonia were done in [52].

C. Leptonic constants

In the static approximation for the heavy quarks the calculation of leptonic constants for the heavy quarkonia with the two-loop accuracy involves the matching of leptonic currents in NRQCD with the currents of full QCD,

$$J_\nu^{\rm QCD}=\bar{Q}\gamma_\nu Q,\qquad \mathcal{J}_\nu^{\rm NRQCD}=\chi^\dagger\sigma_\nu^\perp\phi,$$

with the relativistic quark fields Q and their nonrelativistic two-component limits of antiquark χ and quark ϕ, $\sigma_\nu^\perp=\sigma_\nu-v_\nu(\sigma\cdot v)$, and v is the four velocity of heavy quarkonium, so that

$$J_\nu^{\rm QCD}=\mathcal{K}(\mu_{\rm hard};\mu_{\rm fact})\cdot\mathcal{J}_\nu^{\rm NRQCD},$$

where the scale $\mu_{\rm hard}$ determines the normalization point for the matching of NRQCD with full QCD, while $\mu_{\rm fact}$ refers to the point of perturbative calculations in NRQCD. Using the matching of potential for the static quarks in QCD with the

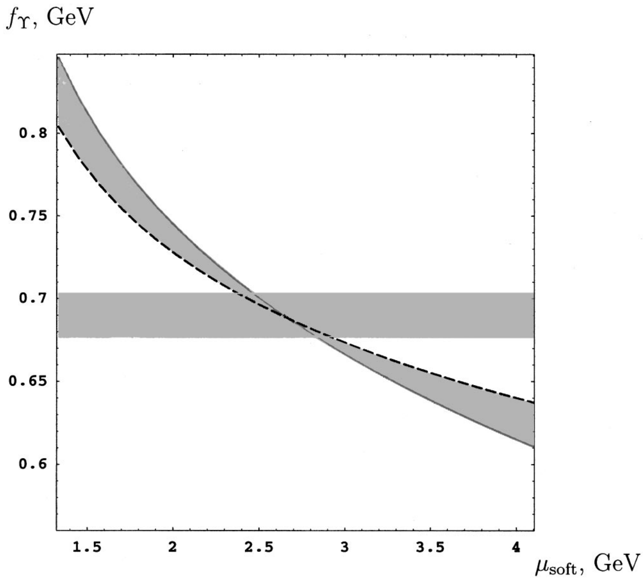

FIG. 7. The value of leptonic constant for the vector ground state of bottomonium vs the soft scale. The dashed line represents the choice of $\mu_{\rm hard}=2m_b$, while the solid line represents $\mu_{\rm hard}=m_b$. The horizontal shaded band gives the experimental limits.

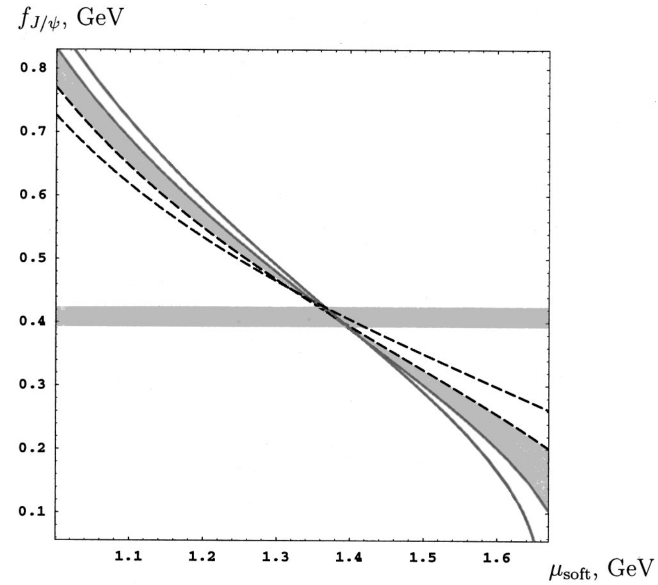

FIG. 8. The value of leptonic constant for the vector ground state of charmonium vs the soft scale. Shaded region is restricted by the dashed line representing the choice of $\mu_{\rm hard}=1.07m_c$ and by the solid line with $\mu_{\rm hard}=0.93m_c$. The horizontal shaded band gives the experimental limits. The additional curves give $\mu_{\rm hard}=1.26m_c$ (dashed line) and $\mu_{\rm hard}=0.87m_c$ (solid line).

two-loop perturbative potential, we argue that the most appropriate choice of scale relevant to the charmonium and bottomonium is

$$\mu_{\rm fact}=\mu_{\rm soft}=1.3\text{–}2 \ \ \text{GeV}. \tag{48}$$

For the heavy quarkonium composed by quarks of the same flavor the Wilson coefficient $\mathcal{K}$ is known up to the two-loop accuracy [53,33,54,55]

$$\mathcal{K}(\mu_{\rm hard};\mu_{\rm fact})=1-\frac{8}{3}\frac{\alpha_s^{\overline{\rm MS}}(\mu_{\rm hard})}{\pi}+\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu_{\rm hard})}{\pi}\right)^2 c_2(\mu_{\rm hard};\mu_{\rm fact}), \tag{49}$$

and c_2 is explicitly given in [54,55]. The additional problem is the convergency of Eq. (49) at the fixed choice of scales. So, putting $\mu_{\rm hard}=(1-2)m_b$ and Eq. (48) we find a good convergency of QCD corrections for the bottomonium and estimate its leptonic constant defined by

$$\langle 0|J_\nu^{\rm QCD}|\Upsilon,\lambda\rangle=\epsilon_\nu^\lambda f_\Upsilon M_\Upsilon,$$

where λ denotes the polarization of vector state ϵ_ν, so that

$$f_\Upsilon=685\pm 30 \ \ \text{MeV},$$

while the experimental value is equal to $f_\Upsilon^{\rm exp}=690\pm 13$ MeV [9].

As we can see in Fig. 7 the variation of hard scale in broad limits leads to the existence of a stable point, where the result is slowly sensitive to such variation. The stability occurs at $\mu_{\rm soft}\approx 2.6$ GeV, where the perturbative potential is still close to the potential of static quarks at the distances characteristic for the 1S level of $\overline{b}b$.

The estimate of leptonic constant for the charmonium J/ψ is more sensitive to the choice of factorization scale. Indeed, the size of this system, $\langle r_{\overline{c}c(1S)}\rangle\approx 0.42$ fm, makes more strict constraints on $\mu_{\rm fact}\approx 1.3$–1.5 GeV, since at higher scales the perturbative potential significantly deviates from the potential of static quarks in QCD in the region of bound $\overline{c}c$ states, while at lower scales the perturbative potential in two loops does not match the QCD potential in all of the form. Another problem is the energy shift $\delta V(\mu)=1.0$–1.2 GeV, which essentially renormalizes the pole mass of charmed quark, $m_c^{\rm pole}=1.968$–2.068 GeV. This shift does not perturb the mass of the ground state, but it is significant for the value of wave function at the origin. So, following the well-adjusted scaling relation for the leptonic constants [56], we put $P(\mu)=\kappa\Psi(0)m_c^{\rm pole}(\mu)/m_c$ and use it in the calculations of the leptonic constant.[12] We get

$$f_{J/\psi}=400\pm 35 \ \ \text{MeV},$$

to compare with the experimental value $f_{J/\psi}^{\rm exp}=409\pm 15$ MeV.

In Fig. 8 we see that again the stability point can be reached in the variation of $\mu_{\rm hadr}$ at a reasonable value of $\mu_{\rm soft}\approx 1.35$ GeV. However, the stability takes place in the narrow region of $\mu_{\rm hadr}$ close to the charm quark mass.

[12]Solving the Schrödinger equation with the shifted masses and potential, we check that this mass dependence of wave function is valid with the accuracy better than 6%, so we put $\kappa=0.95$.

TABLE IV. The ratios of leptonic constants for the heavy quarkonia as predicted in the present paper (K^2O) in comparison with the experimental data.

$f^2_{\psi(nS)}/f^2_{\psi}$	QCD, (K^2O)	Exp.	$f^2_{\Upsilon(nS)}/f^2_{\Upsilon}$	QCD (K^2O)	Exp.
$2S$	0.55	0.48 ± 0.07	$2S$	0.47	0.47 ± 0.03
$3S$	0.32	0.25 ± 0.06	$3S$	0.34	0.36 ± 0.02

At present, the matching condition for the heavy quarkonium composed by the quarks of different flavors, $\bar{b}c$, is known to one loop, only [56,57]. So, for the pseudoscalar state we have

$$\mathcal{K}(\mu_{\rm hard};\mu_{\rm fact})=1-\frac{\alpha_s^{\overline{\rm MS}}(\mu_{\rm hard})}{\pi}\left(2-\frac{m_b-m_c}{m_b+m_c}\ln\frac{m_b}{m_c}\right), \tag{50}$$

which is independent of the factorization scale. The matching of perturbative potential to the one-loop accuracy with the QCD potential of static quarks at $r\sim0.3$–0.4 fm relevant to the ground state of B_c meson [58], is rather questionable, since the deviation in the forms of potentials is quite sizable. In addition we have to pose $\mu_{\rm fact}=\mu_{\rm hard}$, because we cannot distinguish these scales, while the nonzero anomalous dimension to two loops is not taken into account. Nevertheless, we can put $\mu_{\rm hard}=1.3$–1.8 GeV and neglect δV, which is beyond the actual control in the one-loop accuracy. Indeed, as we see in Fig. 6 the one loop value of energy shift for the matching of perturbative and QCD potentials is quite small at the large virtualities about 2 GeV, and it can be neglected, while at smaller virtualities the form of perturbative potential is close to that given by QCD only in the short range of distances $r=0.1$–0.25 fm, hence, the results on the matching are not reliable for extracting the heavy quark masses from the parameters of bound states. So, we estimate

$$f_{B_c}=400\pm45 \text{ MeV},$$

to compare with the estimates in the SR, where $f^{\rm SR}_{B_c}=400\pm25$ MeV [56,59].

Finally, we present the ratios of leptonic constants for the excited nS levels of $\bar{b}b$ and $\bar{c}c$ in Table IV in comparison with the experimental data. We see that the predictions are in good agreement with the measured values. For completeness, we also predict the constant of $2S$ level in the $\bar{b}c$ system

$$f_{B_c(2S)}=280\pm50 \text{ MeV},$$

which agrees with the scaling relation [56].

Thus, we have analyzed the estimates following from the potential of static quarks in QCD for the masses of quarks and heavy quarkonia as well as for the leptonic constants, and found both in good agreement with the experimental data available and the consistency with the QCD sum rules.

IV. CONCLUSIONS

We have derived the potential of static heavy quarks in QCD on the base of known limits at short and long distances: the asymptotic freedom to the three-loop accuracy, and the confinement regime. The inputs of potential are the coefficients of perturbative β function, the matching of $\overline{\rm MS}$ scheme with the V scheme of potential, the normalization of running coupling constant of QCD at $\mu^2=m_Z^2$, and the slope of Regge trajectories, determining the linear term in the potential. Thus, the approach by Buchmüller and Tye has been modified in accordance with the current status of perturbative calculations.

In the static limit the two-loop improvement of Coulomb potential results in the significant correction to the β function for the effective charge, $\Delta\beta/\beta\sim10\%$ as shown in Fig. 3. This correction is important for the determination of critical values of charge, i.e., the value in the intermediate region between the perturbative and nonperturbative regimes. Moreover, the two-loop matching condition and the three-loop running of coupling constant normalized by the data at the high energy of m_Z determine the region of energetic scale for changing the regimes mentioned above. This scale strongly correlates with the data on the mass spectra of heavy quarkonia. So, it is connected with the splitting of masses between the $1S$ and $2S$ levels. We stress that the consistent consideration of two-loop improvement gives the appropriate value of effective Coulomb coupling constant as it was fitted in the Cornell model of potential. This is achieved in the present paper in contrast with the one-loop consideration by Buchmüller and Tye, who found the value of $\Lambda_{\rm QCD}$ inconsistent with the current normalization at high energies. So, the two-loop improvement gives the correct normalization of effective Coulomb exchange at the distances characteristic for the average separation between the heavy quarks inside the heavy quarkonium and determines the deviations at short distances $r<0.08$ fm (see Fig. 2), which is important in the calculations of leptonic constants related with the wave functions at the origin.

Other corrections to the potential of heavy quarks are connected with the finite mass effects and cannot be treated in the framework of static approximation. For example, the spin-dependent forces, relativistic corrections, and specific non-Abelian potential terms[13] in the heavy quarkonium should be taken in the analysis of mass spectra. A magnitude of leading nonstatic corrections can be evaluated by the characteristic shifts of levels due to the hyperfine splitting of S-wave levels in the heavy quarkonia.[14] So, we conservatively evaluate the uncertainty of heavy quark mass analysis $\delta m\simeq80$ MeV.

Thus, the non-Abelian term of potential α_s^2/r^2, say, has the factors in the form of $1/m_Q$, and it is equal to zero in the static limit $m_Q\to\infty$, while the uncertainty in the heavy quark

[13]They have the form of α_s^2/r^2 with the factor given by the inverse heavy quark masses.

[14]The splitting is about 100 MeV or less.

masses due to the omission of such terms is estimated in the paragraph above. Formally, if we consider the perturbation theory for the calculation of bound state levels in the heavy quarkonium with the Coulomb functions taken as the leading approximation, which is not a scope of our consideration, then the mentioned non-Abelian potential contributes in the same order in α_s as the two-loop corrections to the matching of perturbative static potential $\sim\alpha_s^4$, since the averaging of $1/r^2$ results in the $\alpha_s^2 m_Q^2$ factor. However, the two-loop effects are important for the consistent consideration of static potential and the high energy normalization, i.e., these corrections are significant in the running of effective charge in the potential from the high energies to the scale relevant to the heavy quark bound states even in the static limit, while the nonstatic contributions can be consistently neglected in the numerical analysis. We see that our consideration is consistent in the static approximation, which we have addressed in the present paper.

The matching of two-loop perturbative potential with the QCD potential of static quarks has been performed to get estimates of heavy quark masses, which can be compared with the results of QCD sum rules. Good agreement between two approaches has been found.

The recent determinations of heavy quark masses in Refs. [32–34] were done in the framework of QCD sum rules, which is a systematic approach, indeed. It is based on the separation of the short-distance region from the nonperturbative effects at some values of parameters defining the scheme of calculations in the sum rules. In this approach the nonperturbative terms are given in the form of quark-gluon condensates contributing with corresponding Wilson coefficients calculated at short distances, as was shown in [37], a numerical contribution of gluon condensate term in the sum rules is negligibly small in comparison with the perturbative part. However, it would be incorrect to think that these explicit contributions suppressed in some region of parameters are the only terms caused by the nonperturbative infrared dynamics of QCD. Indeed, neglecting the condensate terms, we find that the perturbative correlators suffer from the renormalon ambiguity, which implies that the perturbative expansion in series of α_s is asymptotic, and the summation of series depends on the method used. The physical reason for such divergency and ambiguity is the infrared singularity in the QCD coupling constant. This singularity is regularized by introducing the threshold mass parameters free of renormalon. Such an approach is independent of any assumptions on the gluon condensate, since generally the pole mass renormalon and the gluon condensates are different issues.

The perturbative pole mass used in the QCD sum rules is not a well defined quantity, and some relevant quantities are introduced in Refs. [32–34]. These quantities are constructed from the perturbative pole mass of heavy quark with specific infrared subtractions, which are treated independently of the quark-gluon condensates. These constructions are author dependent, though the authors of subtracted masses gave some physical motivations, which are more or less strict, but justified. These infrared subtractions imply the introduction of infrared regulators.

In the present paper the unified β function for the effective charge in the potential is considered, and its definition supposes the infrared stability. Thus, we see that the analysis of heavy quark masses in both the QCD sum rules and potential approach involves the consideration of relevant effects caused by the infrared dynamics of QCD, although the explicit constructive procedures are certainly different, but they have similar inherent uncertainties.

The calculated mass spectra of heavy quarkonia and the leptonic constants of vector nS levels are in agreement with the measured values. The characteristics of the B_c meson have been predicted.

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